University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 3

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Reading:
105 Notes 4.1-4.6, 5.1-5.3.
Hand & Finch 1.4, 2.1-2.9, 1.10-1.11

1.
Use the calculus of variations to show that the shortest distance between two points in three-dimensional space is a straight line.
Solution:
The distance from point 1 to point 2 along a curve $(x, y(x), z(x))$ is just

$$\begin{aligned}\ell &= \int_1^2 \sqrt{dx^2 + dy^2 + dz^2} \\ &= \int_1^2 \sqrt{1 + y'^2 + z'^2}\, dx\end{aligned}$$

where y' means dy/dx.

The variation of this integral is zero (and so the integral is an extremum) when the Euler-Lagrange equations

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}$$

and

$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = \frac{\partial f}{\partial y}$$

are satisfied. In our case, these equations are

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

and

$$\frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

So the two terms in the parentheses are constants. Call them p and q respectively. Then our two equations become

$$p^2 + (p^2 - 1)y'^2 + p^2 z'^2 = 0$$

and

$$q^2 + q^2 y'^2 + (q^2 - 1)z'^2 = 0$$

Solving for y' and z' gives

$$y' = \frac{p}{\sqrt{1 - p^2 - q^2}}\ , \quad z' = \frac{q}{\sqrt{1 - p^2 - q^2}}$$

So y' and z' are just constants. But that means that $y(x)$ and $z(x)$ are just ordinary linear equations. So our curve is a straight line.

2.
Use the calculus of variations to obtain the function $\phi(\theta)$ describing the "great circle" path of minimum length on the surface of a sphere. This path connects spherical polar coordinates (θ_1, ϕ_1) with (θ_2, ϕ_2), in the general case where $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$. Leave your answer in the form of an integral equation. [*Hint*: consider θ to be a "label" (like time t), and ϕ to be a coordinate (like $q(t)$).]
Solution:
In spherical coordinates, an infinitesimal displacement $\vec{dl}$ can be written as:

$$\vec{dl} = dr\ \hat{r} + r\ d\theta\ \hat{\theta} + r\ \sin\theta\ d\phi\ \hat{\phi}$$

In the context of this problem, however, we will be considering a path on the surface of a sphere, so $dr = 0$, and $r = R =$ constant. Also,

$$d\phi = \frac{d\phi}{d\theta} d\theta = \phi' d\theta\ .$$

Hence, the total path length on the surface of the sphere between θ_1 and θ_2 is

$$\begin{aligned}l &= \int_{\theta_1}^{\theta_2} \left|\vec{dl}\right| \\ &= R \int_{\theta_1}^{\theta_2} d\theta \sqrt{1 + \sin^2\theta\ \phi'^2}\end{aligned}$$

Calling the integrand $\mathcal{L}(\phi,\phi',\theta)$, we apply the Euler-Lagrange equation:

$$\frac{d}{d\theta}\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{\partial \mathcal{L}}{\partial \phi}$$
$$\frac{d}{d\theta}\left(\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}}\right) = 0$$
$$\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}} = c$$

where c is a constant of integration. Solving for ϕ' yields

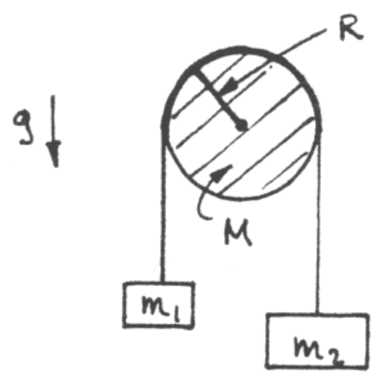
$$\phi' = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}\,.$$

Both sides of this expression can be integrated from θ_1 to θ to obtain

$$\phi(\theta) = \int_{\theta_1}^{\theta} \frac{c}{\sin\theta'\sqrt{\sin^2\theta' - c^2}}\,d\theta' + \phi_1$$

where c is chosen so that $\phi(\theta_2) = \phi_2$.

3.
Set up and solve the Euler-Lagrange equation for the Atwood machine, released from rest. (Two weights $m_1 < m_2$ are suspended via a massless string that is supported by a pulley in the form of a disk of radius R and mass M. The string moves without slipping on the pulley.)



Use the height $y(t)$ of the smaller mass as the generalized coordinate.

Solution:
Let's choose as our generalized coordinate y, the distance of mass m_1 above its starting point. Then $-y$ is the distance of m_2 above its starting point, and $\theta = y/R$ is the angle through which the wheel has rotated. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - V\\ &= \tfrac{1}{2}(m_1+m_2)\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2 - (m_1-m_2)gy\end{aligned}$$

The moment of inertia of a disc is $I = \frac{1}{2}MR^2$, so

$$\mathcal{L} = \tfrac{1}{2}(m_1+m_2)\dot{y}^2 + \tfrac{1}{4}M\dot{y}^2 - (m_1-m_2)gy$$

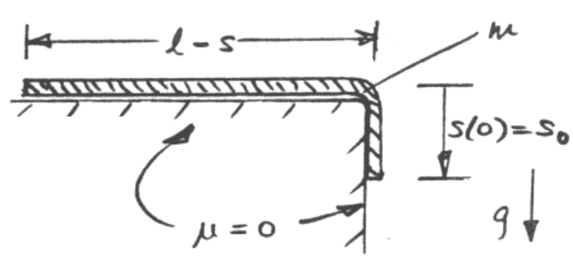
Let's define $\mu = m_1 + m_2 + \frac{1}{2}M$ and $\Delta m = m_1 - m_2$. Then $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \Delta m g y$, and the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$ gives

$$\mu\ddot{y} = -\Delta m g$$

The acceleration is constant: $a = -\frac{\Delta m}{\mu}g$. So

$$y = -\frac{1}{2}\frac{\Delta m}{\mu}gt^2 = \left(\frac{m_2 - m_1}{2m_1 + 2m_2 + M}\right)gt^2$$

4.
A chain of mass m and length l lies on a frictionless table. Initially the chain is at rest, with a length $s = s_0$ of the chain hanging off the table's end. This causes the chain to fall off the table. The part of the chain that remains on the table is straight, not coiled.



Using the Euler-Lagrange equation with s as the generalized coordinate, calculate the motion of the chain (before it falls off completely). Assume that the chain remains in contact with the corner and end of the table as shown (even though this in fact is true only for the early part of the motion).

Solution:

Let's take the zero point of the potential energy to be the surface of the table. That way, only that portion of the chain which is hanging contributes to the potential energy. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m\dot{s}^2 + \frac{mg}{2l}s^2\end{aligned}$$

Applying the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) &= \frac{\partial \mathcal{L}}{\partial s} \\ \ddot{s} &= \frac{g}{l}s\end{aligned}$$

The general solution of which is

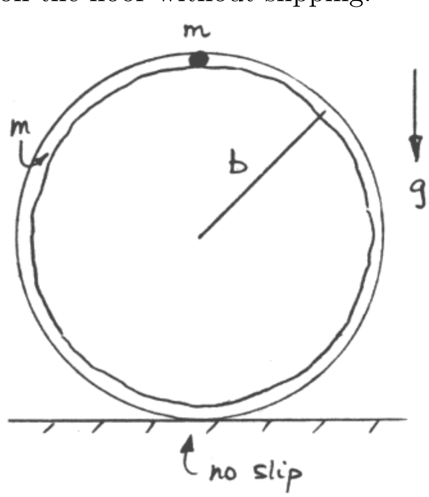
$$s(t) = A\sinh\sqrt{\frac{g}{l}}t + B\cosh\sqrt{\frac{g}{l}}t$$

where A and B are constants. We determine A and B using our boundary conditions, s(0) $= s_o$ and $\dot{s}(0) = 0$, to get

$$s(t) = s_o\cosh\sqrt{\frac{g}{l}}t$$

5.

A bead of mass m moves inside a thin hoop-shaped pipe of average radius b, also of mass m. The pipe has a frictionless interior, so that the bead moves freely within the circumference of the hoop. But the coefficient of friction between the floor and the pipe's exterior is large, so that the hoop rolls on the floor without slipping.



The bead is released from rest at the top of the hoop. When the bead has fallen halfway to the floor, how far to the side will the hoop have moved?

Solution:

Let's take as our generalized coordinates θ, the angle of the bead from the top of the hoop, and x, the sideways distance that the center of the hoop has moved from its starting point. We'll assume that the bead falls clockwise, for which we'll define θ to be positive, and we'll take x to be positive in the *left*-hand direction. The kinetic energy is

$$\begin{aligned}T &= T_{\text{hoop}} + T_{\text{bead}} \\ T_{\text{hoop}} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mb^2\left(\frac{\dot{x}}{b}\right)^2 \\ &= m\dot{x}^2 \\ T_{\text{bead}} &= \frac{1}{2}m\left((-\dot{x} + b\dot{\theta}\cos\theta)^2 + (-b\dot{\theta}\sin\theta)^2\right) \\ &= \frac{m}{2}\left(\dot{x}^2 + b^2\dot{\theta}^2 - 2b\dot{x}\dot{\theta}\cos\theta\right) \\ T &= m\left(\frac{3}{2}\dot{x}^2 + \frac{1}{2}b^2\dot{\theta}^2 - b\dot{x}\dot{\theta}\cos\theta\right)\end{aligned}$$

If we take the zero of the potential energy to be at the center of the hoop, then only the bead has potential energy.

$$U = mgb\cos\theta$$

Thus the Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= m\left(\frac{3}{2}\dot{x}^2 + \frac{1}{2}b^2\dot{\theta}^2 - b\cos\theta(mg + \dot{x}\dot{\theta})\right)\end{aligned}$$

Applying the Euler-Lagrange equation to the coordinate x:

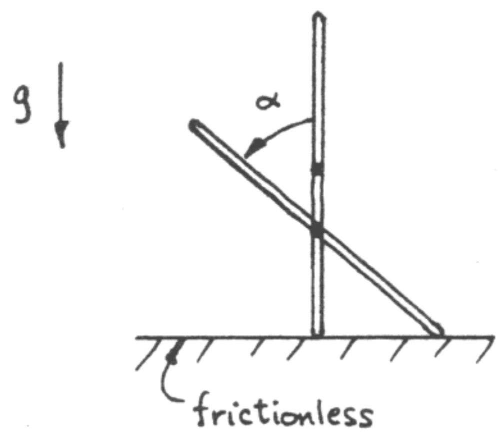
$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\ \frac{d}{dt}\left(3\dot{x} - b\dot{\theta}\cos\theta\right) &= 0 \\ 3\dot{x} - b\dot{\theta}\cos\theta &= c_1 \quad \text{(a constant)}\end{aligned}$$

By the initial conditions, $\dot{x}(0) = \dot{\theta}(0) = 0$, $c_1 = 0$.

$$\begin{aligned} 3\dot{x} - b\dot{\theta}\cos\theta &= 0 \\ \frac{d}{dt}\left(3x - b\sin\theta\right) &= 0 \\ 3x - b\sin\theta &= c_2 \quad (c_2 = 0 \text{ also}) \\ x &= \frac{1}{3}b\sin\theta \end{aligned}$$

At $\theta = \frac{\pi}{2}$, $x = \frac{b}{3}$. The hoop is displaced in the opposite direction from that of the bead.

6.
At $t = 0$, a thin uniform stick, resting on a frictionless floor, is erect and motionless. Let α represent the angle it makes with the vertical (initially $\alpha = 0$).



(**a**)
Use the Euler-Lagrange equation to obtain an equation relating $\ddot{\alpha}$ to α and $\dot{\alpha}$.
Solution:
The height of the center of mass is $y = \frac{1}{2}l\cos\alpha$, so the kinetic energy (including both translational and rotational terms) is $T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\alpha}^2$. The moment of inertia of a thin stick is $I = \frac{1}{12}ml^2$, so $T = \frac{1}{24}ml^2\dot{\alpha}^2(1 + 3\sin^2\alpha)$. The potential energy is $V = mgy = \frac{1}{2}mgl\cos\alpha$. So $\mathcal{L}$ is

$$\mathcal{L} = \tfrac{1}{24}ml^2\dot{\alpha}^2(1 + 3\sin^2\alpha) - \tfrac{1}{2}mgl\cos\alpha$$

The Euler-Lagrange equation for α is

$$\frac{1}{4}ml^2\dot{\alpha}^2\sin\alpha\cos\alpha + \frac{1}{2}mgl\sin\alpha =$$

$$\frac{1}{12}ml^2\left(\ddot{\alpha}(1 + 3\sin^2\alpha) + 6\dot{\alpha}^2\sin\alpha\cos\alpha\right)$$

Solving this for $\ddot{\alpha}$ yields

$$\ddot{\alpha} = \frac{6\frac{g}{l}\sin\alpha - 3\dot{\alpha}^2\sin\alpha\cos\alpha}{1 + 3\sin^2\alpha}$$

(**b**)
Because the floor is frictionless, total mechanical energy is conserved in this problem. Use this fact to relate $\dot{\alpha}$ to α.
Solution:

$$\begin{aligned} E &= \frac{1}{24}ml^2\dot{\alpha}^2(1 + 3\sin^2\alpha) + \frac{1}{2}mgl\cos\alpha \\ &= \frac{1}{2}mgl = E_{\text{initial}} \\ \dot{\alpha}^2 &= \frac{12g}{l}\frac{1 - \cos\alpha}{1 + 3\sin^2\alpha} \end{aligned}$$

(**c**)
Use the result of (**b**) to eliminate $\dot{\alpha}$ from your answer to (**a**), thereby obtaining an equation relating $\ddot{\alpha}$ to α alone. This equation should be valid for all values of α.
Solution:
Inserting the answer from (**b**) into the result of (**a**), and simplifying, yields

$$\ddot{\alpha} = \frac{6g}{l}\sin\alpha\left(\frac{1 + 3(1 - \cos\alpha)^2}{\left(1 + 3\sin^2\alpha\right)^2}\right)$$

(**d**)
In the limit $\alpha \ll 1$, solve the result of (**c**) for the motion $\alpha(t)$.
Solution:
When $\alpha \ll 1$, we keep only terms to first order in α. In that limit, the expression from (**c**) becomes:

$$\ddot{\alpha} \approx \frac{6g}{l}\alpha$$

If we take our initial conditions to be $\alpha(0) = \alpha_o \ll 1$ and $\dot{\alpha}(0) = 0$, then the solution to this is

$$\alpha(t) = \alpha_o\cosh\sqrt{\frac{6g}{l}}t$$

7.
Continue to consider the stick in the previous problem. Use the method of Lagrange undetermined multipliers to find the force of constraint exerted by the floor on the stick, at the instant before the side of the stick impacts the floor.
Solution:
Let v be the height of the bottom point of the stick from the floor. Because the stick's lower end is always in contact with the ground, we have the constraint that $v = 0$. The height of the CM is now $y = v + \frac{l}{2}\cos\alpha$.

$$\begin{aligned}
T &= \frac{m}{2}\dot{y}^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\alpha}^2 \\
&= \frac{m}{8}l^2\dot{\alpha}^2\left(\frac{1}{3} + \sin^2\alpha\right) + \frac{m}{2}\dot{v}^2 - \frac{ml}{2}\dot{v}\dot{\alpha}\sin\alpha \\
V &= mgy \\
&= mg\left(v + \frac{1}{2}l\cos\alpha\right)
\end{aligned}$$

From the equation of constraint, we know that $g_v = 1$. Applying the Euler-Lagrange equation to the v coordinate:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{v}}\right) &= \frac{\partial\mathcal{L}}{\partial v} + g_v\lambda \\
\frac{d}{dt}\left(m\dot{v} - \frac{ml}{2}\dot{\alpha}\sin\alpha\right) &= -mg + F_c \\
m\ddot{v} - \frac{ml}{2}\left(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha\right) &= -mg + F_c\ ,
\end{aligned}$$

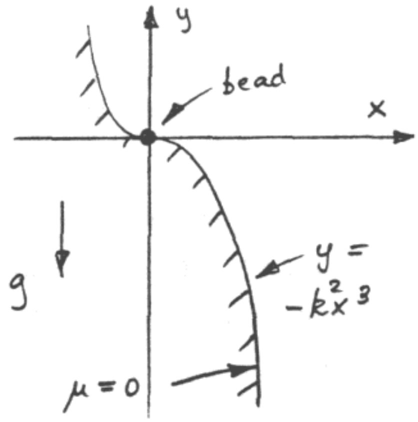
where $F_c = g_v\lambda$ is the generalized force of constraint. Using the fact that $\ddot{v} = \dot{v} = v = 0$, and the expression for $\ddot{\alpha}$ from **6(c)**, yields

$$\begin{aligned}
F_c &= mg - \frac{ml}{2}\ddot{\alpha}|_{\alpha=\frac{\pi}{2}} \\
&= mg - \frac{ml}{2}\left(\frac{3}{2}\frac{g}{l}\right) \\
&= \frac{1}{4}mg \quad \text{(upward)}
\end{aligned}$$

8.
A bead moves under the influence of gravity on a frictionless surface described by

$$y = -k^2x^3\ ,$$

where k is a constant, and x and y are the horizontal and vertical coordinates.



The bead is released from rest at the origin. Use the method of Lagrange undetermined multipliers to solve for the coordinate $x = x_0$ at which it leaves the surface.
Solution:

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy
\end{aligned}$$

Constraint:

$$\begin{aligned}
y + k^2x^3 &= 0 \quad \text{(Eq. 3)} \\
dy + 3k^2x^2dx &= 0 \\
\Rightarrow\ g_y \ = \ 1,\ g_x &= \ 3k^2x^2
\end{aligned}$$

Applying the Euler-Lagrange equations:

$$\begin{aligned}
m\ddot{y} &= -mg + \lambda \quad \text{(Eq. 1)} \\
m\ddot{x} &= 3k^2x^2\lambda \quad\ \ \text{(Eq. 2)}
\end{aligned}$$

Use (Eq. 3) to eliminate $\ddot{y}$ from (Eq. 1):

$$\begin{aligned}
\dot{y} &= -3k^2x^2\dot{x} \\
\ddot{y} &= -6k^2x\dot{x}^2 - 3k^2x^2\ddot{x} \\
3k^2x(x\ddot{x} + 2\dot{x}^2) &= g - \frac{\lambda}{m} \quad \text{(Eq. 1}'\text{)}
\end{aligned}$$

Use (Eq. 2) to eliminate $\ddot{x}$ from (Eq. 1′):

$$3k^2x\left(3k^2x^3\frac{\lambda}{m} + 2\dot{x}^2\right) = g - \frac{\lambda}{m} \quad \text{(Eq. 1}''\text{)}$$

The bead loses contact when $\lambda = 0$. Set $\lambda = 0$ in (Eq. $1''$), to get:

$$6k^2 x\dot{x}^2|_{x_1=\text{breakaway}} = g \quad \text{(Eq. 4)}$$

Use energy conservation, $\dot{x}^2 + \dot{y}^2 = -2gy$, and (Eq. 3) to solve for x in terms of $\dot{x}$:

$$\dot{x}^2 + \left(-3k^2x^2\dot{x}\right)^2 = -2g(-k^2x^3)$$
$$\dot{x}^2 = \frac{2gk^2x^3}{1+9k^4x^4} \quad \text{(Eq. 5)}$$

Substitute (Eq. 5) in (Eq. 4):

$$6k^2x\frac{2gk^2x^3}{1+9k^4x^4}|_{x_1} = g$$
$$x_1 = \frac{1}{k3^{\frac{1}{4}}}$$